Competition between dynamic and thermal relaxation in non-equilibrium critical spin systems above the critical point

Competition between dynamic and thermal relaxation in non-equilibrium critical spin systems above the critical point

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Abstract
We study the long-time behaviour and the spatial correlations of a simple
ferromagnetic spin system whose kinetics is governed by a thermal bath
with a time-dependent temperature which is characterized by a given external
relaxation time τ. Exact results are obtained in the framework of the spherical
model in d dimensions. In the paramagnetic phase, the long-time kinetics is
shown to depend crucially on the ratio between τ and the internal equilibration
time $\tau_{\rm eq}$. If $\tau \lesssim \tau_{\rm eq}$, the model relaxes rapidly towards an equilibrium state but
there appear transient and spatially oscillating contributions in the spin–spin
correlation function. On the other hand, if $\tau \gg \tau_{\rm eq}$ the system is clamped and
its time evolution may be delayed with respect to that of the heat bath. For
waiting times s such that $\tau \gg s \gg \tau_{\rm eq}$, a quasi-stationary state is found where
the fluctuation–dissipation theorem does not hold.

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1. Introduction

Non-equilibrium systems generally display specific features which are absent from their
corresponding thermodynamically equilibrated counterparts. For example, consider a many-
body system contained in a thermostat and prepared in some initial state. Then assume that
the temperature of the heat bath is rapidly quenched to a different temperature T_1 and the
system is then allowed to evolve freely, with the temperature fixed at T_1. In this setting, the

system under study may show ageing effects, which come from the dynamical breaking of time-translation invariance, see [1–4] for reviews. Although ageing has been known, for a very long time, to occur in glassy systems, similar effects have more recently also been observed for non-disordered spin systems. In simple ferromagnets such as the Ising model, ageing effects have been studied through the long-time behaviour of two-time correlation and response functions, see [5] and references therein. In addition, it has been proposed recently that the well-known dynamical scale invariance, associated with the ageing phenomenon (see e.g. [1, 2]) in simple ferromagnets, may be generalized towards a larger group of local dynamical scale transformations [6]. Among other results this theory leads to an explicit prediction for the time-dependent thermoremanent magnetization which has been confirmed in several distinct models [7–11].

However, current theoretical studies generally assume that the quench is infinitely rapid, or in other words that the final temperature T_1 is reached immediately. In real experiments, this condition is often far from being satisfied, see e.g. [12, 13]. It is therefore interesting to investigate the behaviour of a system coupled to a bath with a time-dependent temperature $T(t)$. Here, we shall study the long-time behaviour of a system initially prepared in a fully disordered state and brought at time $t = 0$ into contact with a thermal bath which evolves from an initial temperature T_0 to a final temperature T_1 in a finite or infinite interval of time. In this paper, we shall study the following form of a time-dependent temperature:

$$T(t) = T_1 + (T_0 - T_1) \exp(-t/\tau) \tag{1.1}$$

where τ is a given relaxation time which sets an external time scale $\tau_{\text{ext}} = \tau$. Certainly, the long-time behaviour of a statistical system will depend, besides the external time scale τ, on the internal time scale(s) $\tau_{\text{int}}(T)$ which arise from the internal self-organization of the system, governed by the microscopic interactions. Two basic cases should be expected. First, if $\tau_{\text{int}} \gg \tau$, the system simply relaxes towards equilibrium at the final temperature T_1. On the one hand, if $\tau_{\text{int}} \ll \tau$, the system is said to be *clamped* at the external temperature $T(t)$ and its time history will depend on the properties of the external heat bath. While for times $t \gg \tau \gg \tau_{\text{int}}$ the system should have simply relaxed back to equilibrium, new effects might be expected to occur in the regime $\tau \gg t \gg \tau_{\text{int}}$ and this may allow us to distinguish between these two cases. The time-dependence as specified in equation (1.1) should contain the basic ingredients which are needed for a conceptual understanding of this kind of phenomenon[6].

One of the central questions is whether/when under the conditions just described the system may evolve towards thermodynamic equilibrium. A convenient way to measure the distance of a system from equilibrium is through the fluctuation–dissipation ratio [15, 16]

$$X(t, s) = T R(t, s) \left(\frac{\partial C(t, s)}{\partial s} \right)^{-1} \tag{1.2}$$

where $C(t, s)$ and $R(t, s)$ are the two-time autocorrelation and autoresponse functions, respectively (see section 2 for the precise definitions). At equilibrium, the fluctuation–dissipation theorem states that $X(t, s) = 1$. Therefore, an important question is under which physical conditions can equilibrium be reached. This question has received much attention in glasses, see [17, 18] for recent theoretical and [12, 19, 20] for recent experimental examples. Understanding this point must come before the more complicated questions relating to an eventual ageing behaviour can be addressed.

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[6] Originally, we became interested in this problem through the consideration of the fragmentation of microscopic systems such as atomic nuclei where it has often been assumed that the detected fragments are at thermodynamic equilibrium and fixed temperature [14]. However, such descriptions do not take care of the time evolution of these systems which expand in space over a finite time interval and cool down before they are detected.

In order to obtain explicit analytical results, we shall study the exactly solvable kinetic spherical model with a time-dependent bath temperature, to be defined precisely in section 2. For constant temperatures, this model has been studied in great detail in the past, either in the context of continuum field theories [9, 21–25] or else in the form of a lattice model [8, 10, 26–30]. It is well established that the spherical model, despite its technical simplicity, still contains the main physical features encountered in other systems (such as the Ising model) which might be considered to be closer to the experimental reality but which no longer permit an exact analytical solution. In particular, the results found from the spherical model in $d < 4$ space dimensions are known to be different from the predictions of mean-field theory. In this paper, we shall be concerned exclusively with the question of under what conditions a stationary or quasi-stationary state of the model may be considered to be an equilibrium state in the sense that $X(t, s) = 1$. This question can be studied by restricting attention to the high-temperature phase such that the system is brought into contact with a heat bath at the initial temperature T_0. The heat bath then cools down to the final temperature T_1 such that both are above the critical temperature, namely $T_0 > T_1 > T_c > 0$ and the model remains in the disordered high-temperature phase. In this phase, there is no ageing behaviour to be expected. We leave the question of a possible ageing behaviour in systems coupled to a time-dependent bath for future work.

The content of the paper is as follows. In section 2 we introduce the model and obtain the formal exact solution. The equal-time correlator is analysed in section 3 and we discuss several new non-trivial time and length scales which arise. Two-time quantities and the fluctuation–dissipation ratio are studied in section 4 and the results are generalized to include the clamped case in section 5. Section 6 presents our conclusions.

2. Model and formalism

We begin by recalling the definition of the kinetic spherical model, using the formalism as exposed in [8, 26, 28]. We consider a system of time-dependent classical spin variables $S_x(t)$ located on the sites x of a d-dimensional hypercubic lattice. They may take arbitrary real values subject only to the spherical constraint

$$\sum_x S_x(t)^2 = \mathcal{N} \tag{2.1}$$

where $\mathcal{N}$ is the number of sites of the lattice. The spherical model Hamiltonian reads

$$\mathcal{H} = -J \sum_{\langle x, y \rangle} S_x(t) S_y(t) \tag{2.2}$$

where the sum extends over nearest-neighbour pairs only. In the following we choose units such that $J = 1$. The system is supposed to be translation invariant in all directions. The kinetics is assumed to be described in terms of a Langevin equation

$$\frac{\mathrm{d} S_x(t)}{\mathrm{d} t} = \sum_{y(x)} S_y(t) - (2d + \mathfrak{z}(t)) S_x(t) + \eta_x(t) \tag{2.3}$$

where the sum over y extends over the nearest neighbours of x and $\eta_x(t)$ corresponds to a stochastic force which describes the action of an environment which lies outside the nearest-neighbour range. Physically, this means that the model is assumed to be immersed in a heat bath. The resulting forces are supposed to be Gaussian and thus to be characterized by an ensemble average which is zero, namely $\langle \eta_x(t) \rangle = 0$, and a second moment which reads

$$\langle \eta_x(t) \eta_y(t') \rangle = 2T(t) \delta_{x,y} \delta(t - t'). \tag{2.4}$$

At time $t = 0$ the system is brought into contact with the heat bath. We assume in this work that the temperature of the heat bath decreases from T_0 to T_1 with a characteristic relaxation time τ according to

$$T(t) = T_1 + (T_0 - T_1) \exp(-t/\tau). \tag{2.5}$$

The choice of this temporal behaviour is dictated by computational simplicity. In addition, as we shall see, it already contains the basic effects which will also be present for more general thermal histories. For sufficiently large values of the initial bath temperature T_0, the typical correlation lengths are of the order of a lattice constant or less. Then for all practical purposes, the system is effectively uncorrelated. Finally, the function $\mathfrak{z}(t)$ is fixed by the spherical constraint (2.1) and has to be determined. In order to do this, we follow the standard procedure of replacing the spherical constraint by its mean value. It can be shown that this does not change the results if the infinite-system limit $\mathcal{N} \to \infty$ is taken before the long-time limit $t \to \infty$ [30].

By a Fourier transformation

$$\tilde{f}(\boldsymbol{q}) = \sum_{\boldsymbol{r}} f_{\boldsymbol{r}} \, \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \qquad f_{\boldsymbol{r}} = (2\pi)^{-d} \int_{\mathcal{B}} \mathrm{d}\boldsymbol{q}\, \tilde{f}(\boldsymbol{q})\, \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \tag{2.6}$$

where the integral is taken over the first Brillouin zone $\mathcal{B}$, the Fourier-transformed spin variable $\tilde{S}(\boldsymbol{q}, t)$ becomes

$$\tilde{S}(\boldsymbol{q}, t) = \frac{\mathrm{e}^{-\omega(\boldsymbol{q})t}}{\sqrt{g(t)}} \left[\tilde{S}(\boldsymbol{q}, 0) + \int_0^t \mathrm{d}t'\, \mathrm{e}^{\omega(\boldsymbol{q})t'} \sqrt{g(t')} \tilde{\eta}(\boldsymbol{q}, t') \right] \tag{2.7}$$

with the dispersion relation

$$\omega(\boldsymbol{q}) = 2 \sum_{i=1}^{d} (1 - \cos(q_i)) \tag{2.8}$$

and we have also defined

$$g(t) = \exp\left(2 \int_0^t \mathrm{d}t'\, \mathfrak{z}(t') \right). \tag{2.9}$$

Clearly, the time-dependence of $\tilde{S}(\boldsymbol{q}, t)$ and any correlators will be given in terms of the function $g = g(t)$.

We now derive the expressions for the correlators and response functions for an arbitrary thermal history $T = T(t)$. We begin with the equal-time spin–spin correlation function

$$C_{\boldsymbol{x},\boldsymbol{y}}(t) = C_{\boldsymbol{x}-\boldsymbol{y}}(t) = \langle S_{\boldsymbol{x}}(t) S_{\boldsymbol{y}}(t) \rangle. \tag{2.10}$$

We obtain the Fourier transform $\tilde{C}(\boldsymbol{q}, t)$ from

$$\langle \tilde{S}(\boldsymbol{q}, t) \tilde{S}(\boldsymbol{q}', t') \rangle = (2\pi)^d \delta(\boldsymbol{q} + \boldsymbol{q}') \tilde{C}(\boldsymbol{q}, t) \tag{2.11}$$

and immediately find

$$\tilde{C}(\boldsymbol{q}, t) = \frac{\mathrm{e}^{-2\omega(\boldsymbol{q})t}}{g(t)} \left[\tilde{C}(\boldsymbol{q}, 0) + 2 \int_0^t \mathrm{d}t'\, T(t')\, \mathrm{e}^{2\omega(\boldsymbol{q})t'} g(t') \right]. \tag{2.12}$$

If we assume uncorrelated initial conditions[7]

$$C_{\boldsymbol{x},\boldsymbol{y}}(0) = \delta_{\boldsymbol{x}-\boldsymbol{y},\mathbf{0}} \tag{2.13}$$

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[7] The treatment of correlated initial conditions, following the lines of [8], presents no additional difficulty.

as we shall always do in the following, we have $\tilde{C}(\boldsymbol{q}, 0) = 1$. Because of the spherical constraint (2.2) and spatial translation invariance, the autocorrelator must satisfy

$$C_{\boldsymbol{0}}(t) = \int_{\mathcal{B}} \mathrm{d}\boldsymbol{q} \, \tilde{C}(\boldsymbol{q}, t) = \langle S_{\boldsymbol{x}}(t)^2 \rangle = 1. \tag{2.14}$$

This in turn fixes $\mathfrak{z}(t)$ or via (2.9) the function $g(t)$ as the solution of a Volterra integral equation

$$g(t) = f(t) + 2 \int_0^t \mathrm{d}t' \, T(t') f(t - t') g(t') \tag{2.15}$$

where

$$f(t) = \frac{1}{(2\pi)^d} \int_{\mathcal{B}} \mathrm{d}\boldsymbol{q} \, \mathrm{e}^{-2\omega(q)t} = (\mathrm{e}^{-4t} I_0(4t))^d \tag{2.16}$$

and I_0 is a modified Bessel function [31]. Once we have found the solution of equation (2.15), the correlation function can be obtained.

Before doing this, we now give the expression for the two-time correlation function $C_{\boldsymbol{x}-\boldsymbol{y}}(t, s) = \langle S_{\boldsymbol{x}}(t) S_{\boldsymbol{y}}(s) \rangle$ and the two-time response function $R_{\boldsymbol{x}}(t, s)$. The calculation follows entirely the standard lines [8, 26, 28] and we merely quote the result. In Fourier space

$$\tilde{C}(\boldsymbol{q}, t, s) = \tilde{C}(\boldsymbol{q}, s) \, \mathrm{e}^{-\omega(q)(t-s)} \sqrt{\frac{g(s)}{g(t)}} \tag{2.17}$$

and where $g = g(t)$ is the solution of equation (2.15). Similarly, the response function is obtained in the usual way [8, 23, 24, 26, 28] by adding a small magnetic field term $\delta\mathcal{H} = -\sum_{\boldsymbol{x}} h_{\boldsymbol{x}}(t) S_{\boldsymbol{x}}(t)$ to the Hamiltonian. We easily find in Fourier space

$$\tilde{R}(\boldsymbol{q}, t, s) = \left. \frac{\delta \langle \tilde{S}(\boldsymbol{q}, t) \rangle}{\delta \tilde{h}(\boldsymbol{q}, s)} \right|_{h_r = 0} = \mathrm{e}^{-\omega(q)(t-s)} \sqrt{\frac{g(s)}{g(t)}}. \tag{2.18}$$

From these expressions, the autocorrelation function $C(t, s) = C_{\boldsymbol{0}}(t, s)$ and the autoresponse function $R(t, s) = R_{\boldsymbol{0}}(t, s)$ can be obtained by integrating over the momentum $\boldsymbol{q}$.

Summarizing, the physically interesting correlation and response functions are given by equations (2.12), (2.17) and (2.18) together with the constraint equation (2.15), for any time-dependent temperature $T = T(t)$. This constitutes the main result of the general formalism.

In order to solve equation (2.15) explicitly, we now use the specific form (2.5) of the time-dependent temperature $T = T(t)$. Through a Laplace transformation

$$\bar{f}(p) = \int_0^\infty \mathrm{d}t f(t) \, \mathrm{e}^{-pt} \tag{2.19}$$

equation (2.15) is transformed into a linear difference equation

$$\bar{g}(p) = \bar{f}(p) + 2T_1 \bar{f}(p) \bar{g}(p) + 2(T_0 - T_1) \bar{f}(p) \bar{g}(p + 1/\tau). \tag{2.20}$$

The analytic solution of this equation allows us to study the behaviour of the correlation and response functions defined above. Two limit cases correspond to constant temperatures and have been analysed in detail in the literature [28]. First, in the limit $\tau \to \infty$, we have a constant temperature $T = T_0$ and the final temperature T_1 is never reached. Second, the limit $\tau \to 0$ corresponds to an infinitely rapid quench to the constant end temperature $T = T_1$. We are interested in the behaviour between these two extremes. The solution $\bar{g}(p)$ of equation (2.20) can be cast in the form

$$\bar{g}(p) = \sum_{n=0}^{\infty} (T_0 - T_1)^n \bar{g}_n(p) \tag{2.21}$$

where

$$\bar{g}_n(p) = 2^n \prod_{k=0}^{n} \bar{g}^{(0)} \left(T_1, p + \frac{k}{\tau} \right). \tag{2.22}$$

Here $\bar{g}^{(0)}(T_1, p)$ is the solution as given by Godrèche and Luck [28] for a fixed temperature $T = T_1$

$$\bar{g}^{(0)}(T_1, p) = \frac{\bar{f}(p)}{1 - 2T_1 \bar{f}(p)}. \tag{2.23}$$

In the next section we shall turn to a detailed analysis of the properties of the physical observables coming from this solution.

3. The equal-time correlator

We now analyse the long-time behaviour of the equal-time spin–spin correlation function $C_x(t)$ or rather its Fourier transform $\tilde{C}(q, t)$. Throughout this paper, we shall restrict ourselves to the situation where *both* the initial temperature T_0 and the final temperature T_1 are above the critical point and the systems are cooled from T_0 to T_1, namely

$$T_0 > T_1 > T_c. \tag{3.1}$$

The equilibrium critical temperature is given by $T_c = (2\bar{f}(0))^{-1}$, see [25, 26, 28], and is nonzero for $d > 2$. As we have seen in the last section, the time-dependence of the correlators follows from the form of the function $g(t)$. In turn, the behaviour of $g(t)$ can be described in terms of the singularities of its Laplace transform $\bar{g}(p)$. From equations (2.21) and (2.22), these singularities are entirely given in terms of the singularities of the constant-temperature solution $\bar{g}^{(0)}(T_1, p)$ of equation (2.23).

In the high-temperature phase, the singularities of $g^{(0)}(T_1, t)$ can be analysed as follows [28]. Consider the function

$$\bar{f}(p) = \frac{1}{(2\pi)^d} \int_{\mathcal{B}} \mathrm{d}q \, \frac{1}{p + 2\omega(q)} \tag{3.2}$$

which is monotonically decreasing with p. Furthermore, $\bar{f}(p)$ is analytic in the complex p-plane except for a cut in the interval $-8d \leqslant p \leqslant 0$. On the other hand, if $T_1 \geqslant T_c$, the function $\bar{g}^{(0)}(T_1, p)$ has a simple pole at a value p_0 given by $\bar{f}(p_0) = 1/(2T_1)$. This translates into an exponential long-time behaviour of $g^{(0)}(T_1, t) \sim \exp(t/\tau_{\mathrm{eq}})$ which defines an equilibration time

$$\tau_{\mathrm{eq}} := \tau_{\mathrm{eq}}(T_1) = \tau_0 = \frac{1}{p_0}. \tag{3.3}$$

Now, for a time-dependent temperature $T = T(t)$, the only possible singularities of $\bar{g}(p)$ are those of $\bar{g}^{(0)}(T_1, p + k/\tau)$ and therefore the singularities of $\bar{g}(p)$ with a positive real part are simple poles and occur at

$$p_k := p_0 - \frac{k}{\tau} \tag{3.4}$$

with $k = 0, 1, 2, \ldots, k_{\mathrm{max}}$. Here k_{max} is the largest integer such that $p_{k_{\mathrm{max}}}$ is still positive. For $k > k_{\mathrm{max}}$, the singularities merely lead to exponentially decreasing corrections in $g(t)$ and do not contribute to the leading long-time behaviour we are interested in. The sequence of poles p_k leads to the following sequence of relaxation times:

$$\frac{1}{\tau_k(T_1)} = \frac{1}{\tau_{\mathrm{eq}}(T_1)} - \frac{k}{\tau}. \tag{3.5}$$

The quantities p_k are positive for $0 \leqslant k \leqslant k_{\max}$ and negative for $k > k_{\max}$. Hence for t sufficiently large, i.e. $t \gtrsim \tau_{\mathrm{eq}}(T_1)$, the leading contributions to $g(t)$ are of the form

$$g(t) \simeq \sum_{k=0}^{k_{\max}} G_k \exp(t/\tau_k) \tag{3.6}$$

where

$$G_k = \gamma \sum_{n=0}^{\infty} 2^n (T_0 - T_1)^n \prod_{j=0, j \neq k}^{n} \bar{g}^{(0)}(T_1, p_0 + (j-k)/\tau) \tag{3.7}$$

and in addition

$$\gamma = -\bar{f}(p_0)^2 / \bar{f}'(p_0) \tag{3.8}$$

is a positive constant and the prime denotes the derivative.

Using expression (3.6) the spin–spin correlation function given by equation (2.12) takes the following form:

$$\tilde{C}(\boldsymbol{q}, t) \simeq T_1 \left(\sum_{k=0}^{k_{\max}} G_k \exp(-kt/\tau) \right)^{-1} \left[\frac{G_0}{\omega(\boldsymbol{q}) + \lambda_{\mathrm{eq}}^{-2}} + \sum_{\ell=1}^{k_{\max}} \frac{H_\ell}{\omega(\boldsymbol{q}) + \lambda_\ell^{-2}} \, \mathrm{e}^{-\ell t/\tau} \right.$$

$$\left. + \frac{T_0 - T_1}{T_1} \frac{G_{k_{\max}}}{\omega(\boldsymbol{q}) - \lambda_{k_{\max}+1}^{-2}} \, \mathrm{e}^{-(k_{\max}+1)t/\tau} \right] + O(\mathrm{e}^{-2\omega(\boldsymbol{q})t}) \tag{3.9}$$

provided $t \gtrsim \tau_{\mathrm{eq}}(T_1)$ and where for all $0 \leqslant \ell \leqslant k_{\max} + 1$ we have defined

$$\lambda_\ell = +\sqrt{2|\tau_\ell|} = \left| \frac{2\tau_{\mathrm{eq}}(T_1)\tau}{\tau - \ell\tau_{\mathrm{eq}}(T_1)} \right|^{1/2}. \tag{3.10}$$

In the following, we shall use the relation $\lambda_{\mathrm{eq}} = \lambda_0 = \sqrt{2\tau_{\mathrm{eq}}}$. In addition, we also used the abbreviation

$$H_k = G_k + \frac{T_0 - T_1}{T_1} G_{k-1}. \tag{3.11}$$

In order to understand the behaviour of $\tilde{C}(\boldsymbol{q}, t)$, we note that the first term in equation (3.9), which formally corresponds to $\ell = 0$, gives the stationary contribution. In addition, there is at least one additional transient term which will disappear in the limit $t \gg \tau$. The number of transient terms which are present depends on the value of $k_{\max} = k_{\max}(T_1, \tau)$. We now discuss the various physical cases which can arise.

3.1. First-mode contribution—short thermal relaxation times

The simplest case arises if $\tau_{\mathrm{eq}} \geqslant \tau$, that is the bath relaxes faster than the system itself. Then $p_0 - \frac{1}{\tau} \leqslant 0$, that is $k_{\max} = 0$ and we have in equation (3.9) $\ell = 0$ as the only contributing mode. Therefore (assuming $t > \tau$ and $t \gg \tau_{\mathrm{eq}}(T_1)$)

$$\tilde{C}(\boldsymbol{q}, t) \simeq T_1 \left(\frac{1}{\omega(\boldsymbol{q}) + \lambda_{\mathrm{eq}}^{-2}(T_1)} + \frac{(T_0 - T_1)}{T_1} \frac{\mathrm{e}^{-t/\tau}}{\omega(\boldsymbol{q}) - \lambda_1^{-2}} \right) + O(\mathrm{e}^{-2\omega(\boldsymbol{q})t - t/\tau}). \tag{3.12}$$

For a physical understanding, it might be more appealing to consider the correlator in real space. In the 1D case, the result is particularly simple and in the large-separation limit $|\boldsymbol{x}| \gg 1$ becomes

$$C_{\boldsymbol{x}}(t) \simeq T_1 \left(\lambda_0 \, \mathrm{e}^{-|\boldsymbol{x}|/\lambda_0} - \frac{T_0 - T_1}{T_1} \lambda_1 \sin\left(\frac{|\boldsymbol{x}|}{\lambda_1} \right) \mathrm{e}^{-t/\tau} \right) + O(\mathrm{e}^{-\boldsymbol{x}^2/t} \, \mathrm{e}^{-t/\tau}). \tag{3.13}$$
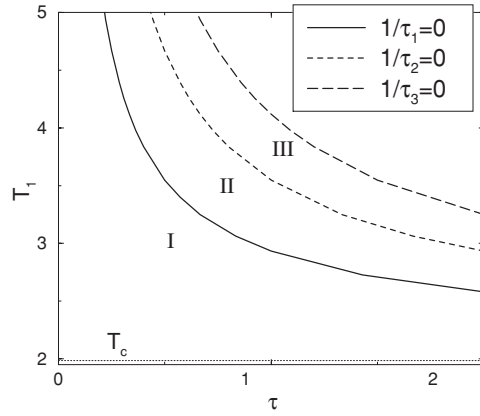
Figure 1. Kinetic phase diagram of the 3D spherical model with a time-dependent temperature as given by equation (2.5). The lines shown are the loci where the relaxation times τ_ℓ, $\ell = 1, 2, 3$, diverge. The dotted line marked T_c gives the equilibrium critical point and the regions I, II and III are those with $k_{max} = 0, 1, 2$, respectively.

Therefore, the correlator is the sum of a stationary and spatially homogeneous term and a transient term which in addition shows spatial oscillations of wavelength λ_1. We point out that λ_1 diverges when the external time scale τ approaches the internal equilibration time τ_{eq} from below, see equation (3.10). In figure 1 we show the location of the line $\tau_{eq} = \tau$ as the curve $1/\tau_1 = 0$ in the (T_1, τ)-plane. The region I in figure 1 corresponds to the case $k_{max} = 0$.

For larger values of τ we go over to a multimode regime which we discuss below.

3.2. Multimode contributions

If $k_{max} > 0$ several transient modes contribute to $\tilde{C}(q, t)$. Their contributions are quite different, however. The propagator arising in all terms is of the form

$$\frac{1}{\omega(q) \pm \lambda_k^{-2}} \overset{q \to 0}{\simeq} \frac{1}{q^2 \pm \lambda_k^{-2}} \tag{3.14}$$

with a positive sign for $k \leqslant k_{max}$ and a negative sign for $k = k_{max} + 1$. This leads to a different spatial behaviour, in analogy with equation (3.13) found for the case $k_{max} = 0$ discussed above. The first k_{max} terms all lead to spatially decaying contributions to $C_x(t)$ in the large-$|x|$ limit, where the characteristic length scale is given by λ_ℓ with $\ell = 0, 1, \ldots, k_{max}$. With the exception of the $\ell = 0$ contribution, all terms are transient and decay with a relaxation time given by τ/ℓ. In contrast, the last term, which is also transient, gives rises to a spatially oscillating term with a wavelength $\lambda_{osc} := \lambda_{k_{max}+1}$.

The behaviour of the system in the first few regimes is illustrated in figure 1. For a fixed final temperature T_1 and increasing values of the external relaxation time τ, the system goes over from region I with only a single mode (since $k_{max} = 0$) to region II with two modes, later to region III with three modes and so on.

The behaviour of the various correlations and wavelengths is illustrated in figure 2 for the 3D case and for $\tau_{eq} = 0.4$, that is $T_1 = 3.8363\ldots$. First, we note that the characteristic length scale λ_{eq} associated with the only non-transient term does *not* depend on the externally imposed time scale τ. Second, the thick solid lines show the dependence of the oscillation wavelength $\lambda_{osc} = \lambda_{k_{max}+1}$ on τ. If we start with a small value of τ in region I and then increase τ, λ_{osc} increases and finally diverges when $\tau = \tau_{eq}$, that is, when the external time scale
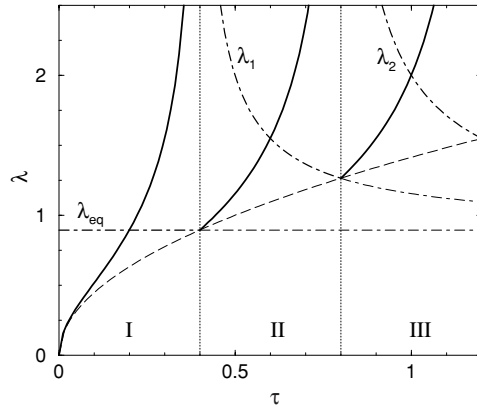
Figure 2. Dependence of some correlations and wavelengths λ_ℓ on the external relaxation time τ, at fixed final temperature $T_1 = 3.8363\ldots$ in the 3D spherical model. The regions I, II, III correspond to the cases $k_{max} = 0, 1, 2$, respectively. The thick solid lines give the oscillation wavelength $\lambda_{osc} = \lambda_{k_{max}+1}$. The dash-dotted lines give the equilibration length scale $\lambda_{eq} = \lambda_0$ (the equilibration time $\tau_{eq} \sim \lambda_{eq}^2$) and the first additional length scales λ_1 and λ_2. The dashed line corresponds to the curve $\lambda = \sqrt{2\tau}$, see the text.

becomes equal to the internal equilibration time scale. If τ is now increased further, we enter region II. At this point, the previous oscillating mode transforms into a spatially monotonic one which simply gives another contribution which decreases as a function of $|\boldsymbol{x}|$ and in addition, a new oscillating mode with a *finite* wavelength appears. This new mode arises since one of the modes which in region I leads to an exponentially decaying contribution to $g(t)$ transforms itself into an, albeit non-leading, exponentially growing contribution. Furthermore, the characteristic length of the old oscillating mode (denoted by λ_1 in figure 2) now decreases as a function of τ. We now increase τ further until the oscillation wavelength λ_{osc} diverges again and we therefore go over from region II into region III. At this transition point, we observe that the value of λ_1 agrees with that of the new oscillation wavelength for region III. Phenomenologically, this looks as if that mode would split into two modes, one giving rise to a spatially oscillating contribution to $C_x(t)$ and the other to a spatially decaying term. This splitting occurs along the line $\lambda = \sqrt{2\tau}$ and in particular, it follows that the oscillation length always satisfies $\lambda_{osc} \geqslant \sqrt{2\tau}$.

4. Two-time correlation and response functions

Using the expression of the function $g(t)$ from equation (3.6), we can formally compute the two-time observables we are interested in, that is the two-time autocorrelation function and the autoresponse function defined in equations (2.17) and (2.18). In particular, a study of these two functions allows us to calculate the fluctuation–dissipation ratio $X(t, s)$. A deviation of this ratio from unity should signal a departure from thermodynamic equilibrium. For a sudden quench to a final temperature T_1 this question has already been studied [28] and it was shown that indeed $X(t, s) = 1$ in the long-time limit in the high-temperature phase when $T_1 > T_c$. Here, with the additional complication of a time-dependent heat bath temperature, we must first define an instantaneous time-dependent temperature of the system in order to make the criterion of the fluctuation–dissipation ratio applicable. We choose the equilibrium part $\tilde{C}_{eq}$ of the single-time correlator as the physical quantity which measures a temperature.

Consider for a moment the case of a fixed temperature T. The equilibrium part behaves for small momenta (or large distances) as

$$\lim_{q \to 0} \tilde{C}_{\text{eq}}(q) = \lim_{q \to 0} \frac{T}{\omega(q) + \lambda_{\text{eq}}^{-2}} = 2\tau_{\text{eq}}T \qquad (4.1)$$

and depends on T in a known way. Therefore, we use the extension of this relation to *define*, for a time-dependent bath temperature $T(t)$ varying according to equation (2.5), an effective temperature $T_{\text{eff}}(s)$ of the spin system. Taking the explicit form (3.9) of the single-time propagator $\tilde{C}(q, s)$ we set

$$\lim_{q \to 0} \tilde{C}(q, s) = 2\tau_{\text{eq}}(T_1)T_{\text{eff}}(s) \qquad (4.2)$$

which is the defining equation for the effective time-dependent temperature $T_{\text{eff}}(s)$. It remains to be seen whether this T_{eff} can play the role of a physical equilibrium temperature.

This can be seen by means of a comparison with the results obtained from an instantaneous quench towards T_1, hence $\tau = 0$. For clarity, we shall denote by $A^{(0)}$ the result obtained for an observable A, after a sudden quench with $\tau = 0$ to the final temperature T_1. The two-time autocorrelation function $C^{(0)}(t, s)$ is explicitly given in equation (2.58) of [28]. In particular, it is known that [28] $\lim_{q \to 0} \tilde{C}^{(0)}(q, s) = 2\tau_{\text{eq}}(T_1)T_1$.

We now consider the two-time autocorrelation function $C(t, s)$. From section 2, we have

$$C(t, s) = \sqrt{\frac{g(s)}{g(t)}} \int \mathrm{d}q \, \tilde{C}(q, s) \, \mathrm{e}^{-\omega(q)(t-s)}$$

$$\simeq \sqrt{\frac{g(s)}{g(t)}} \int \mathrm{d}q \, \tilde{C}(0, s) \, \mathrm{e}^{-q^2(t-s)}$$

$$= \sqrt{\frac{g(s)}{g(t)}} \int \mathrm{d}q \, 2\tau_{\text{eq}}(T_1)T_{\text{eff}}(s) \, \mathrm{e}^{-q^2(t-s)}$$

$$= \sqrt{\frac{g(s)}{g(t)}} \int \mathrm{d}q \frac{T_{\text{eff}}(s)}{T_1} \tilde{C}^{(0)}(0, s) \, \mathrm{e}^{-q^2(t-s)}$$

$$\simeq \frac{T_{\text{eff}}(s)}{T_1} k^{(0)}(t, s) C^{(0)}(t, s). \qquad (4.3)$$

In the second and the fifth lines we used the fact that we are in the regime $t - s \gg 1$, in combination with the fact that in this limit, only the low-q behaviour of $C(t, s)$ and of $C^{(0)}(t, s)$ really contributes. In the fourth line, the definition of $T_{\text{eff}}(s)$ was used. For brevity, we have set

$$k^{(0)}(t, s) = \sqrt{\frac{g(s)/g^{(0)}(s)}{g(t)/g^{(0)}(t)}} \qquad (4.4)$$

where the Laplace transform of $g^{(0)}(t)$ was given in (2.23). The exact autoresponse function is given by

$$R(t, s) = \int \mathrm{d}q \, \tilde{R}(q, t, s) = f\left(\frac{t - s}{2}\right) \sqrt{\frac{g(s)}{g(t)}} = R^{(0)}(t, s) k^{(0)}(t, s). \qquad (4.5)$$

We now ask whether the quasi-equilibrium temperature $T_{\text{eff}}(s)$ so defined is really the temperature of a system in equilibrium. A convenient way to test this is through the fluctuation–dissipation ratio

$$X(t, s) = T_{\text{eff}}(s) R(t, s) \left(\frac{\partial C(t, s)}{\partial s} \right)^{-1}. \tag{4.6}$$

An equilibrium situation can only be recovered if $X(t, s) = 1$. We now bring this expression into a simpler form as follows:

$$
\begin{aligned}
\frac{1}{X(t, s)} &= \frac{\partial C(t, s)/\partial s}{T_{\text{eff}}(s) R(t, s)} \\
&= \frac{1}{T_{\text{eff}}(s)} \frac{C(t, s)}{R(t, s)} \frac{\partial C(t, s)/\partial s}{C(t, s)} \\
&\simeq \frac{1}{T_{\text{eff}}(s)} \frac{C^{(0)}(t, s) T_{\text{eff}}(s) T_1^{-1} k^{(0)}(t, s)}{R^{(0)}(t, s) k^{(0)}(t, s)} \frac{\partial}{\partial s} \ln C(t, s) \\
&= \frac{1}{T_1} \frac{C^{(0)}(t, s)}{R^{(0)}(t, s)} \frac{\partial}{\partial s} \ln C(t, s) \\
&\simeq \left(\frac{\partial}{\partial s} \ln C(t, s) \right) \left(\frac{\partial}{\partial s} \ln C^{(0)}(t, s) \right)^{-1}.
\end{aligned}
\tag{4.7}
$$

In performing this calculation, we used in the third line the approximate expressions (4.3) and (4.5) valid for $t - s \gg \tau_{\text{eq}}(T_1)$ and in the last line, we used the fact that the fluctuation–dissipation theorem holds for any instantaneous quench to the final temperature $T_1 > T_c$ [28].

After some algebra, we can write this in a compact form, taking also the $t \to \infty$ limit

$$X(t, s) = \left[1 + \frac{\tau_{\text{eq}}}{\tau} \Phi\left(\frac{s}{\tau} \right) \right]^{-1}. \tag{4.8}$$

Remarkably, this scaling form only depends on the single scaling variable $x = s/\tau > 1$. The associated scaling function reads

$$\Phi(x) = \left(\frac{\partial}{\partial x} \ln \Phi_1(x) + 2 \frac{\partial}{\partial x} \ln \Phi_2(x) \right) \tag{4.9}$$

where explicitly

$$\Phi_1(x) = \sum_{k=0}^{k_{\max}} \frac{G_k}{G_0} e^{-kx} \tag{4.10}$$

$$\Phi_2(x) = T_1 \left[\sum_{\ell=0}^{k_{\max}} G_\ell \, e^{-\ell x} \right]^{-1} \left[G_0 \lambda_{\text{eq}}^2 + \frac{T_0 - T_1}{T_1} G_{k_{\max}} \lambda_{k_{\max}+1}^2 \, e^{-(k_{\max}+1)x} + \sum_{k=1}^{k_{\max}} H_k \lambda_k^2 \, e^{-kx} \right]. \tag{4.11}$$

A particularly simple form is found for $k_{\max} = 0$. Physically, this corresponds to the situation when $\tau \leqslant \tau_{\text{eq}}(T_1)$. Then the fluctuation–dissipation ratio reads

$$X(t, s) = \left[1 + 2 \frac{\tau_{\text{eq}}}{\tau} \left(\frac{\lambda_0^2 T_1}{\lambda_1^2 (T_0 - T_1)} \exp(x) - 1 \right)^{-1} \right]^{-1} \qquad x = \frac{s}{\tau}. \tag{4.12}$$

As in the case of an infinitely rapid quench, the fluctuation–dissipation ratio is unity, up to corrections which vanish exponentially fast in the long-time limit. We can conclude that the system evolves towards thermodynamic equilibrium.

5. The clamped case and the limit $\tau/\tau_{eq}(T_0) \to \infty$

Studying the kinetics with a time-dependent temperature and in the disordered phase, we may identify the following time regimes:

1. $\tau \ll \tau_{eq}(T_1)$. This case formally corresponds to $k_{max} = 0$. Studying the long-time kinetics, we always work in the situation where $t \gg \tau_{eq}(T_1) \gg \tau$. Up to exponentially small corrections, we recover an equilibrium dynamics as found for an infinitely rapid quench [28].

2. $\tau \simeq \tau_{eq}(T_1)$. In this case, we consider $t \gg \tau_{eq}(T_0)$ and also $t \gg \tau$. This regime corresponds to a finite nonzero value of k_{max}. With respect to the situation of an infinitely rapid quench, there are several new exponentially small contributions which arise, as was shown in section 4. As expected, equilibrium dynamics is recovered.

3. $\tau \gg \tau_{eq}(T_1)$. This is called the clamped regime[8] because the microscopic degrees of freedom follow the varying temperature of the external heat bath. Certainly, for times $t \gg \tau \gg \tau_{eq}(T_0)$ we shall simply recover equilibrium dynamics, along the lines of section 4. Because we consider only quenches which reduce the temperature $T_0 > T_1 > T_c$ throughout, we always have $\tau_{eq}(T_1) > \tau_{eq}(T_0)$. On the other hand, new and interesting long-time behaviour occurs in the regime $\tau_{eq}(T_0) \ll t \ll \tau$. In this case, the system is in equilibrium as far as the internal degrees of freedom are concerned, since $t \gg \tau_{eq}(T_0)$, but has not yet equilibrated with respect to the external time scale of the heat bath, since $t \ll \tau$. (A special case of the case under consideration here corresponds to $k_{max} = \infty$.)

We first have to look closer at the limit $\tau \to \infty$. As we have seen above, the central part of the calculation is the solution of the Volterra integral equation (2.15) coming from the spherical constraint. However, for $\tau \to \infty$ and after having performed the Laplace transformation, we may expand in $1/\tau$ and then obtain to leading order an ordinary differential equation

$$\frac{d\bar{g}(p)}{dp} + \tau a(p)\bar{g}(p) = \tau b(p) \tag{5.1}$$

where

$$a(p) = \frac{2T_0\bar{f}(p) - 1}{2(T_0 - T_1)\bar{f}(p)} \qquad b(p) = -\frac{1}{2}\frac{1}{T_0 - T_1}. \tag{5.2}$$

Although this equation may be solved straightforwardly through integrations, it is for our purposes more useful to proceed differently. After all, we are merely interested in the large-time (or small-p) behaviour of the correlators. In addition, because of the $\tau \to \infty$ limit implicit in equation (5.1), we must take the limit in such a way that $\tau a(p)$ remains finite. Therefore, we merely need $a(p)$ in the vicinity of p_c defined by $a(p_c) = 0$.[9] The leading long-time behaviour should then be given by an expansion in powers of $p - p_c$. We write, using (5.2)

$$a(p) \simeq \Delta(p - p_c) + \cdots \qquad \Delta = \frac{2T_0^2}{T_0 - T_1}\left.\frac{d\bar{f}(p)}{dp}\right|_{p=p_c} < 0. \tag{5.3}$$

We look for a solution of (5.1) of the form

$$\bar{g}(p) = \sum_{n=0}^{\infty} \tau^{-n}\bar{g}_n(p) \tag{5.4}$$

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[8] En français: serré; auf deutsch: eingeklemmt.

[9] Physically, one may understand this by noting that in the case of constant temperature T_0, τ is formally infinite and we should have to leading order $\bar{g}(T_0, p) = b(p)/a(p)$. Therefore, the root p_c sets the entire physical behaviour, up to normalization factors.

and we directly obtain

$$\bar{g}_n(p) = \frac{b}{\Delta} \left(\frac{2}{\Delta}\right)^n \frac{\Gamma(n+1/2)}{\Gamma(1/2)} (p-p_c)^{-2n-1}. \tag{5.5}$$

In order to perform the inverse Laplace transformation, we write

$$\bar{g}(p) = \sum_{n=0}^{\infty} \tau^{-n} \int_0^{\infty} dt \, g_n(t) \, e^{-pt} = \sum_{n=0}^{\infty} \int_0^{\infty} dt \frac{b}{\Delta} \left(\frac{t^2}{2\tau\Delta}\right)^n \frac{1}{n!} e^{-(p-p_c)t}. \tag{5.6}$$

At this stage the sum may be performed first and we find

$$g(t) = \sum_{n=0}^{\infty} g_n(t) \, e^{p_c t} = \frac{b}{\Delta} \exp\left(p_c t - \frac{t^2}{\tau_{\text{eff}}^2}\right) \tag{5.7}$$

which is valid provided $t \ll \tau$. In this expression, τ_{eff} is the typical time after which the function g and then the correlators which are closely related to it differ notably from the expressions they take at formally infinite external relaxation time τ. We find

$$\tau_{\text{eff}} = \sqrt{-2\tau\Delta} = \sqrt{2\tau \frac{2T_0^2}{T_0 - T_1} \left|\frac{d\bar{f}(p)}{dp}\right|_{p=p_c}}. \tag{5.8}$$

Remarkably enough, τ_{eff} differs from the external time $\tau_{\text{ext}} = \tau$. Rather, it is the geometric mean between the externally imposed relaxation time scale as measured by τ and an internal time scale, measured by $\tau_{\text{dep}} = 2\Delta(T_0, T_1)$. In the physical observables, τ_{eff} appears as a natural time scale informing on the departure from an effectively constant temperature $T = T_0$. We point out that this new time scale τ_{dep} is distinct from the equilibration time at temperature T_0, namely $\tau_{\text{eq}}(T_0) = 1/p_c$.

For example, the two-time response function becomes

$$R(t, s) = R^{(0)}(t, s) \exp\left(-\frac{(t-s)(t+s)}{4\tau|\Delta|}\right) \tag{5.9}$$

which illustrates again the role of τ_{eff} in the description of the deviation with respect to $R^{(0)}(t, s)$.

We now comment on how these results depend on spatial dimensionality d. From equation (5.8), we directly see that τ_{eff} diverges when T_0 and T_1 get closer to each other. Far from the critical region, we have $\tau_{\text{eff}} \simeq (T_0 - T_1)^{-1/2}$, independently of d. On the other hand, when the system enters the critical region, results depend on whether fluctuations are strong as is the case for $2 < d < 4$ or whether one has mean-field criticality when $d > 4$. In the fluctuation-dominated regime, that is $2 < d < 4$, the first derivative of $\bar{f}(p)$ diverges as p goes to zero. Using the known small-p expansion of $\bar{f}$ [28, 32, 33], we find

$$\tau_{\text{eff}} \simeq \left[\frac{T_1 - T_c}{T_c}\right]^{-\frac{4-d}{2(d-2)}} (T_0 - T_1)^{-1/2}. \tag{5.10}$$

On the other hand, in the mean-field regime $d > 4$, the first derivative of $\bar{f}(p)$ remains bounded even in the neighbourhood of the critical point. Then we simply retain $\tau_{\text{eff}} \simeq (T_0 - T_1)^{-1/2}$.

For $2 < d < 4$, we have thus found a rather counterintuitive effect. Namely, the time scale τ_{eff} up to which the system evolves as if the temperature were fixed at T_0, increases when the final temperature T_1 approaches the critical temperature T_c. This enhancement of τ_{eff} is absent for $d > 4$ and must therefore be related to the presence of strong fluctuation effects close to T_c and below the upper critical dimension.

Finally, we consider the two-time observables and the fluctuation–dissipation ratio. We have just seen that for τ large, the kinetics of the system occurs at an infinitesimally slow rate

for a long time. In contrast to the treatment of section 4, it is therefore useful to compare the time-evolving quantities with those when formally $\tau = \infty$ or, in physical terms, when the temperature remains fixed at T_0. Formally, the calculations can be done by analogy with section 4. First, we use the single-time correlator to define an effective time-evolving temperature $T_{eff}(s)$ from the relationship

$$\lim_{q \to 0} \tilde{C}(q, s) = 2\tau_{eq}(T_0)T_{eff}(s). \tag{5.11}$$

We shall denote by $A^{(\infty)}$ the result obtained for an observable A with a time evolution governed by an infinitely long external time $\tau = \infty$. In particular, we have $\lim_{q \to 0} \tilde{C}^{(\infty)}(q, s) = 2T_0\tau_{eq}(T_0)$. As in section 4, it is easy to see that

$$C(t, s) = \frac{T_{eff}(s)}{T_0}k^{(\infty)}(t, s)C^{(\infty)}(t, s) \qquad k^{(\infty)}(t, s) = \sqrt{\frac{g(s)/g^{(\infty)}(s)}{g(t)/g^{(\infty)}(t)}} \tag{5.12}$$

where $g^{(\infty)}(t)$ is again the solution of the spherical constraint (2.15) but now for $\tau = \infty$ and therefore T_1 is replaced by T_0.

We now want to find $T_{eff}(s)$ explicitly. Using the general expressions for the single-time correlator $\tilde{C}(q, t)$ from section 2, the explicit form of $g(t)$ found above and taking the limit $q \to 0$, we obtain

$$T_{eff}(s) = \frac{\Delta \exp\left(-p_c s + s^2/\tau_{eff}^2\right)}{2\tau_{eq}(T_0)b}\left[1 + \frac{2b}{\Delta}\int_0^s dt_1 \exp\left(p_c t_1 - \frac{t_1^2}{\tau_{eff}^2}\right)(T_1 + (T_0 - T_1)e^{-t_1/\tau})\right]. \tag{5.13}$$

Now, we consider times s which satisfy

$$\frac{s}{|\Delta|} \ll \frac{\tau}{\tau_{eq}(T_0)}. \tag{5.14}$$

From the explicit form of Δ, it is clear that this condition is the easier to satisfy the closer T_0 is to T_1, even if both are far from the critical point T_c. We then obtain

$$T_{eff}(s) = \frac{\Delta e^{-p_c s}}{2b\tau_{eq}(T_0)}\left[1 + \frac{2b}{\Delta}T_1 p_c^{-1} e^{p_c s} + \frac{2b}{\Delta}(T_0 - T_1)\frac{e^{(p_c - 1/\tau)s}}{p_c - 1/\tau} + O(e^{-p_c s})\right]$$

$$= T(s) \tag{5.15}$$

where in the last step we used that $\tau \gg \tau_{eq}(T_0)$ or equivalently that $p_c \gg 1/\tau$, and we recall the definition $\bar{f}(p_c) = 1/T_0$. Therefore, we see that under the stated conditions, the effective temperature is equal to the temperature of the external heat bath. This nicely confirms that our definition of $T_{eff}(s)$ is a natural one.

Physically, this result illustrates the (expected) applicability of the adiabatic approximation to equal-time two-point correlators. Namely, we have seen that in the limit $\tau \gg \tau_{eq}(T_0)$, the effective temperature $T_{eff}(s)$ equals the bath temperature $T(s)$ and the equal-time two-point correlators are those of a quasi-equilibrium system.

Lastly, we examine to what extent this heuristic expectation concerning equal-time correlators carries over to two-time quantities. First, in an analogous manner to section 4, the two-time response function reads

$$R(t, s) = f\left(\frac{t - s}{2}\right)\sqrt{\frac{g(s)}{g(t)}} = R^{(\infty)}(t, s)k^{(\infty)}(t, s). \tag{5.16}$$

Then, we can easily compute the fluctuation–dissipation ratio $X(t, s)$. Formally, it is again defined by (4.6) and we find as before

$$\frac{1}{X(t, s)} = \left(\frac{\partial}{\partial s}\ln C(t, s)\right)\left(\frac{\partial}{\partial s}\ln C^{(\infty)}(t, s)\right)^{-1} \tag{5.17}$$

and this holds provided $t - s \gg \tau_{eq}(T_0)$. In the adiabatic regime just studied, we have

$$C(t, s) = \frac{T(s)}{T_0} \, e^{(s^2 - t^2)/\tau_{eff}^2} C^{(\infty)}(t, s) \tag{5.18}$$

and the fluctuation–dissipation ratio becomes

$$\frac{1}{X(t, s)} = \left(\frac{\partial}{\partial s} \ln \frac{T(s)}{T_0} + \frac{2s}{\tau_{eff}^2} + \frac{1}{2\tau_{eq}(T_0)} \right) 2\tau_{eq}(T_0)$$

$$= 1 - \frac{2s}{\Delta} \frac{\tau_{eq}(T_0)}{\tau} - 2 \frac{\tau_{eq}(T_0)}{\tau} \frac{(T_0 - T_1) \, e^{-s/\tau}}{T_1 + (T_0 - T_1) \, e^{-s/\tau}}$$

$$\simeq 1 - 2 \frac{\tau_{eq}(T_0)}{\tau} \frac{T_0 - T_1}{T_0} \tag{5.19}$$

where the last relation is valid provided $s \ll \tau$ and where condition (5.14) was used.

For an equilibrium system, it is well known that $X(t, s) = 1$ for all times $s < t$ from the fluctuation–dissipation theorem. Here, we find an intermediate regime $\tau_{eq}(T_0) \ll s \ll \tau$ such that $X(t, s)$ reaches a plateau value which is different from unity. Still, the adiabatic approximation remains valid, up to small corrections in s/τ. However, the fact that for $s \ll \tau$ the difference $X(t, s) - 1$ is finite (albeit it may numerically be small) shows that the validity of the fluctuation–dissipation relation is independent of, and more restrictive than, the adiabatic approximation.

Of course, for $s \gg \tau \gg \tau_{eq}(T_0)$ the fluctuation–dissipation theorem will be recovered. In other words, equation (5.19) shows that the equilibration $s \gg \tau_{eq}(T_0)$ with respect to the internal degrees of freedom is already enough to reproduce the fluctuation–dissipation theorem to a good approximation while it becomes exact when equilibration with respect to the bath, $s \gg \tau$, is also achieved.

6. Conclusions

We have studied how the non-equilibrium kinetics of a simple ferromagnet is modified when the system is not instantaneously quenched to a final temperature T_1 but brought into contact with a heat bath starting at an initial temperature T_0 such that the approach towards T_1 is described by a finite external relaxation time τ, according to (1.1). More general temperature histories $T(t)$ of the heat bath can be decomposed into sums of exponentials via Laplace transformations. Then the physically interesting long-time behaviour will be governed by the largest relaxation time arising in the Laplace spectrum. We have studied as an example the exactly solvable spherical model in order to get physical insight into the role of a finite value of τ. We have concentrated on the high-temperature phase such that $T_0 > T_1 > T_c$, generalizing previous studies [25–29] with $\tau = 0$.

Our results can be summarized as follows:

1. If τ is smaller than or at most of the same order of magnitude as the internal relaxation or equilibration time $\tau_{int} = \tau_{eq}$, then for times large compared to τ, τ_{eq} the system approaches exponentially fast the known stationary state, in agreement with what was found previously for $\tau = 0$ [25–29].

   In addition, there are transient contributions to the spin–spin correlation function with spatial oscillations on finite length scales λ. In particular, if $\tau \leqslant \tau_{eq}$, we find

$$\lambda^{-1} \sim \sqrt{\frac{1}{\tau} - \frac{1}{\tau_{eq}}}. \tag{6.1}$$

2. In this case, the stationary state is indeed in thermodynamic equilibrium and is reached exponentially fast for times larger than τ. We have checked this through explicit calculation of the fluctuation–dissipation ratio $X(t, s) \to 1$ for late times.

3. A qualitatively different behaviour may be observed in the clamped case, where $\tau \gg \tau_{eq}$.

   A common physical example with clamped degrees of freedom is a glass. A system in a glass state remains out of equilibrium as attested by the fact that the fluctuation–dissipation limit $X_\infty = \lim_{s\to\infty} \lim_{t\to\infty} X(t, s) \neq 1$ (experimentally, this has been studied recently for the spin glass $CdCr_{1.7}In_{0.3}S_4$ [19] and a colloidal glass [20]). This means that in a glass different temperatures can be defined, depending on the way temperature is measured. These phenomenological properties characteristic of a glass can be partially reproduced in our simple model, even though we did *not* introduce disorder into the Hamiltonian.

   Specifically, we have shown, in the setting of the kinetic spherical model, that in the clamped case

   (a) the effective time-dependent temperature $T_{eff}(t)$ as defined in equation (5.11) from the equal-time spin–spin correlator agrees with the external bath temperature $T(t)$ for times larger than τ_{eq}.

   (b) the time-evolution of two-time observables is governed by the *initial* temperature T_0 for times up to the new time scale (if $2 < d < 4$)

   $$\tau_{eff} \sim \left(\frac{T_1 - T_c}{T_c} \right)^{-(4-d)/(2(d-2))} (T_0 - T_1)^{-1/2} \qquad (6.2)$$

   distinct from both τ_{eq} and τ and much larger than τ if T_1 is close to criticality. Remarkably, $\tau_{eff} = \sqrt{\tau \tau_{dep}}$ is the geometric mean of τ and the purely internal time scale $\tau_{dep} \neq \tau$, see (5.8). Here the temporal evolution of the two-time observables is delayed with respect to the evolution of the bath temperature $T(t)$.

   (c) there is an intermediate regime $\tau_{eq}(T_0) \ll s \ll \tau$ such that the system's internal degrees of freedom are in quasi-equilibrium while equilibrium is not yet achieved with respect to the temporal evolution of the bath. Then we find that the model reaches a quasi-stationary state, which however cannot be a quasi-equilibrium state, since the fluctuation–dissipation ratio

   $$X(t, s) = \left(1 - \frac{2\tau_{eq}(T_0)}{\tau} \frac{T_0 - T_1}{T_0} \right)^{-1} \qquad (6.3)$$

   takes on a value different from (albeit close to) unity in the regime $\tau_{eq}(T_0) \ll s \ll \tau$. In distinction to glasses, the plateau value (6.3) is only reached in the *intermediate* regime $\tau_{eq}(T_0) \ll s \ll \tau$, before the system relaxes to equilibrium for very large waiting times $s \gg \tau$.

We stress that the clamped case arises for τ large. At first sight, one might have naively expected that if the bath temperature $T(t)$ is slowly changed, the system might be able to follow the evolution of the bath through a sequence of equilibrium states. Although our explicit results are in agreement with the adiabatic approximations they also show that the validity of the fluctuation–dissipation theorem depends on more than the validity of the adiabatic picture. This observation might be of particular significance for the interpretation of experimental studies on the time-evolution of cooling systems (our results only hold if both the initial and final temperatures are in the disordered phase). Finally, it remains an open question as to what happens when the ordered phase is entered. We hope to come back to this problem in the future.

While our results were obtained in the framework of the spherical model, we expect that the qualitative results, in particular those for when an equilibrium state can be reached, should be of broader validity. Tests of this expectation in other systems would be of interest.

Acknowledgments

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